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Variational formulation of Chern–Simons theory for arbitrary Lie groups

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Abstract

We show that the Chern–Simons theory for a principal G-bundle P over a three-dimensional manifold, with G an arbitrary Lie group, can be formulated as a variational problem defined by local data on the bundle of connections $C(P)$ of P. By means of the theory of variational problems defined by local data we prove that the Euler–Lagrange operator and the differential of the Poincaré–Cartan form can be intrinsically expressed in terms of the symplectic form and the curvature morphism of $C(P)$. These facts and the theory of the global inverse problem of the Calculus of Variations allow us to prove that there is indeed a global Lagrangian density for these theories. We also prove that every infinitesimal automorphism of P produces in a natural way an infinitesimal symmetry of the variational problem defined by the Chern–Simons theory. We therefore conclude that the algebra of infinitesimal symmetries of these theories is infinite dimensional.

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1. Introduction

In recent years, Chern–Simons theory has received a great deal of attention both from the physics and mathematics communities, since on the one hand it provides a nontrivial

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instance of a topological field theory whereas on the other its quantum observables lead to new topological invariants [8,26].

The ordinary formulation of Chern–Simons theory is carried out in a principal G-bundle $\pi : P \to X$ over a three-dimensional manifold X. Besides this, the group G is always supposed to be connected, simply connected and compact. Under these conditions the bundle $\pi : P \to X$ is trivial. Therefore given a connection ω on P its Chern–Simons form $\mathrm{CS}(\omega)$ can be regarded as a three-form on X. This fact allows one to formulate the Chern–Simons theory for this class of Lie groups as a variational problem by using $\mathrm{CS}(\omega)$ as a Lagrangian density, see [8]. Thus, this formulation covers for instance the case of $SU(n)$-bundles but not that of $U(n)$-bundles.

The task of extending Chern–Simons theory to general compact Lie groups has been undertaken in [7,13]. The techniques employed in these papers are topological rather than differential-geometric. As a result, the Chern–Simons action constructed there is defined at the level of singular cochains and its values are only determined up to integers.

Furthermore, to the best of our knowledge, the case of arbitrary noncompact Lie groups has not been considered in the literature. It follows that for nontrivial principal bundles there is no formulation of Chern–Simons theory in the framework of the calculus of variations.

One of the aims of this paper is to remedy this situation. We will show that in the general case we can formulate the Chern–Simons theory as a variational problem defined by local data on the bundle of connections $C(P) \to X$ of the principal bundle $P \to X$, we will follow the arguments expounded in [20,21]. Indeed the idea that we shall pursue is a rather natural one. Since the principal bundle P is locally trivial, the Chern–Simons form defines on any trivialization a Lagrangian density and hence a variational problem. The problem now is how to “glue” together, in a meaningful way, all these variational problems.

This observation raises the general question as to whether it is possible to give a sensible definition of a variational problem defined by local data consisting of a family of first order local Lagrangian densities. This question is important in its own right and the theory resulting from its solution may be applicable in many other situations. For instance it has been successfully applied to study particles under the action of electromagnetic fields on Riemannian manifolds with nontrivial topology, see [20]. Therefore in this paper we shall begin studying it in general before concentrating on the particular case of Chern–Simons theory.

Indeed we will see that we can give a geometric description of these variational problems defined by local data. In order to accomplish this task we will make use of the geometric formulation [10,12,15,18] of the calculus of variations to treat each one of the variational problems defined by the local Lagrangian densities. Then, the process of piecing together all these variational problems will be analyzed in the framework of the inverse problem of the calculus of variations, see [1,19,22].

It is well known, see [10,12], that the Poincaré–Cartan form plays a prominent role in the geometric formulation of first order variational problems. In fact, the most important concepts of this theory, such as extremals, infinitesimal symmetries, Noether invariants, regularity, Jacobi fields, formal symplectic structure, etc., can be characterized in terms of the differential of the Poincaré–Cartan form.

Therefore, it seems natural to define a restricted class of local variational problems consisting of those local variational problems such that the family of differentials of the local

Poincaré–Cartan forms glue together to define a global differential form which, according to the terminology employed in [10], will be called the (formal) symplectic form. In the same way we will say that these are the local variational problems of symplectic type.

Given a local variational problem, it is well known [1] that there is a cohomological obstruction to the existence of a global Lagrangian density. Furthermore, this obstruction can be represented by a certain De Rham cohomology class which has to be computed on a case by case basis. However, in the case of our local variational problems of symplectic type we will prove that the obstruction is given by the cohomology class of the (formal) symplectic form.

With these results at our disposal, we will prove that the Chern–Simons theory on a principal bundle $P \to X$ can be formulated as a local variational problem of symplectic type on the bundle of connections $C(P) \to X$. In the paper [11] it was shown, for the first time, that $C(P)$ carries a natural symplectic form with values on the adjoint bundle ad $P \to X$. Recently, this symplectic structure and the Hamiltonian structure attached to it have been studied in great detail in [5].

We will show that the Euler–Lagrange operator and the symplectic form of the Chern–Simons local variational problem can be intrinsically expressed in terms of the symplectic form of $C(P)$ and the curvature morphism defined on the first jet bundle of $C(P)$. Moreover, taking into account that $C(P)$ is an affine bundle over X, we will prove the vanishing of the cohomological obstruction to the existence of a global Lagrangian density for these theories. The question of finding such a global Lagrangian density is still work in progress that requires additional research to be carried out and thus will not be addressed any further in this paper. However this result in itself seems to be remarkable and it may open the road for future developments in Chern–Simons theory.

Another aspect of the Chern–Simons local variational problem that we shall treat in this paper is that of infinitesimal symmetries and their associated Noether invariants. Here again we will adopt a broader point of view. That is, we will study this question not only in the case of Chern–Simons theory but for any local variational problem of symplectic type. As we will see, the definition of infinitesimal symmetries is a rather straightforward one. However, the existence of global Noether invariants is a much more delicate one and it depends on the vanishing of a certain cohomological obstruction depending only on the infinitesimal symmetry and the symplectic form of the local variational problem.

With this theory at hand we will prove that the algebra of infinitesimal symmetries of Chern–Simons theory is infinite dimensional. We will see this by showing that every infinitesimal automorphism of the principal bundle $P \to X$ gives rise, in a natural way, to an infinitesimal symmetry. Moreover, taking into account the properties of Hamiltonian structure associated to the natural symplectic form of $C(P)$, we will prove that all these infinitesimal symmetries admit a global Noether invariant and we will give their explicit expressions.

The organization of the paper is as follows. In Section 2, in order to fix our notation, we collect some well known results of the geometric formulation of the calculus of variations. Section 3 is devoted to the definition and study of variational problems defined by local data. We also recall the necessary facts about the cohomological obstructions that appear in the global inverse problem of the calculus of variations. In Section 4 we apply these results to the particular aspects of the formulation of Chern–Simons theory as a variational problem defined by local data. Section 5 deals with the problem of defining infinitesimal symmetries and Noether invariants for local variational problems. The resulting theory is applied in Section 6 to study the infinitesimal symmetries of Chern–Simons theory.

The study carried out in this paper can be extended to manifolds of arbitrary dimension if we replace the Chern–Simons form with a higher order transgression class. We will address this question in future works.

We end this introduction by stating the mathematical conventions that will be assumed in this paper. Manifolds are supposed to be paracompact, connected and C^∞. We will use without explicit mention the Einstein summation convention.

2. Preliminaries on the geometric formulation of the calculus of variations

There exists a well established geometric formulation of the calculus of variations based on jet bundle techniques. This framework has been developed by several authors; the reader may see for instance [10,12,15,18] and the references cited therein.

In order to fix our notation and for the convenience of the reader we proceed in this section to recall several well known facts of this theory that will be used in the rest of the paper. Although we shall be concerned with first order variational problems, that is, the ones defined by Lagrangian densities on the first jet bundle, the structure of the theory requires us to deal with jet bundles of arbitrary order.

Let $\pi : Y \to X$ be a fibered manifold with $\dim X = n$ and $\dim Y = n + m$. For any pair of nonnegative integers $l < k$ we will denote by $\pi^k_l : J^kY \to J^lY$ the natural projection between the corresponding jet bundles of local sections of $\pi : Y \to X$, and $\pi^k : J^kY \to X$ will be the projection obtained as $\pi^k = \pi \circ \pi^k_0$.

It is well known [17,18] that every jet bundle J^kY has a contact structure which allows one to define the so called contact forms of J^kY. These forms can be defined by means of the kth order structure form $\theta^{(k)}$ which is a one-form on J^kY with values on the vertical bundle $VJ^{k-1}Y$ of the projection $\pi^{k-1} : J^{k-1}Y \to X$. On the other hand we have on J^kY the forms which are horizontal with respect to the projection $\pi^k : J^kY \to X$. We will denote by $\Omega^{p,q}_{J^kY} \subset \Omega^{p+q}_{J^kY}$ the sheaf of $(p+q)$-differential forms on J^kY which are p-contact and q-horizontal.

Starting from the exterior differential [18] one constructs two $\mathbb{R}$-derivations of degree 1, the vertical differential d_v and the horizontal differential d_h, which are sheaf morphisms

$$d_v : \Omega^{p,q}_{J^kY} \to (\pi^{k+1}_k)_* \Omega^{p+1,q}_{J^{k+1}Y}, \qquad d_h : \Omega^{p,q}_{J^kY} \to (\pi^{k+1}_k)_* \Omega^{p,q+1}_{J^{k+1}Y},$$

where $(\pi^{k+1}_k)_*$ denotes the direct image of sheaves under the natural projection $\pi^{k+1}_k : J^{k+1}Y \to J^kY$.

A first order Lagrangian density is a horizontal form of top degree on J^1Y. While it may not be globally defined, if we want to associate to it a variational problem then its domain must be of the form $(\pi^1_0)^{-1}(U)$ for some open set $U \subset Y$. Therefore the sheaf $\mathcal{L}ag_Y$ of first order Lagrangians on Y is defined as

$$\mathcal{L}ag_Y = (\pi^1_0)_* \Omega^{0,n}_{J^1Y},$$

where $n = \dim X$.

A central concept in the formulation of the calculus of variations for first order Lagrangians [12,15,18] is played by the Euler–Lagrange sheaf morphism

$$E : \mathcal{L}ag_Y \to (\pi^2_0)_* \Omega^{1,n}_{J^2Y},$$

which is an $\mathbb{R}$-linear sheaf morphism that sends a Lagrangian L to its Euler–Lagrange form $E(L)$ on J^2Y.

Other remarkable geometric objects which are associated to a given Lagrangian L (see [10,12,17]) are the Poincaré–Cartan form Θ_L and the Legendre form Ω_L, both defined on J^1Y. They are related by the following expression:

$$\Theta_L = L - \theta \wedge \Omega_L,$$

where θ is the first order structure form.

There is a well known relationship between all these forms [1,18], the so called “first variation formula”

$$E(L) = (\pi^2_1)^* \, \mathrm{d}\Theta_L + d_v(\theta \wedge \Omega_L). \tag{1}$$

This formula allows one to characterize the extremals of a variational problem in two different ways [1,10,12,18]. A local section s of the projection $\pi : Y \to X$ is critical for the variational problem defined by a Lagrangian density L if and only if it fulfills the Euler–Lagrange equation

$$E(L) \circ j^2 s = 0. \tag{2}$$

Equivalently, s is critical if and only if

$$(i_D \, \mathrm{d}\Theta_L)_{|_{j^1 s}} = 0, \quad \forall\, D \in \mathfrak{X}(J^1 Y). \tag{3}$$

Let us finish this section by recalling the local expressions of the Euler–Lagrange and Poincaré–Cartan forms that will be used later. We fix a coordinate chart $\{x^\alpha, y^i\}$ on $U \subset Y$ adapted to the fibration $\pi : Y \to X$. The Greek indices $\alpha, \beta, \dots$ run from 1 to n and label the coordinates on the base, Latin indices $i, j, \dots$ run from 1 to m and label the fiber coordinates. We have natural charts induced on J^1Y and J^2Y that we denote $\{x^\alpha, y^i, y^i_\alpha\}$ and $\{x^\alpha, y^i, y^i_\alpha, y^i_{\alpha\beta}\}$, respectively. If we take $\eta = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ then we will have $L = \mathcal{L} \cdot \eta$ for some $\mathcal{L} \in C^\infty((\pi^1_0)^{-1}(U))$ and the local expression of the Euler–Lagrange form is

$$E(L) = \left\{ \frac{\partial \mathcal{L}}{\partial y^i} - \frac{\mathrm{d}}{\mathrm{d}x^\alpha} \left(\frac{\partial \mathcal{L}}{\partial y^i_\alpha} \right) \right\} \mathrm{d}y^i \wedge \eta,$$

where $\mathrm{d}/\mathrm{d}x^\alpha$ is the total derivative with respect to x^α on J^2Y. In the same way the local expression of the Poincaré–Cartan form is

$$\Theta_L = L + \frac{\partial \mathcal{L}}{\partial y^i_\alpha} (\mathrm{d}y^i - y^i_\beta \, \mathrm{d}x^\beta) \wedge i_{\partial/\partial x^\alpha} \eta.$$

3. Variational problems defined by local data

Local variational problems appear naturally in the study of the inverse problem of the Calculus of Variations, which deals with the question of deciding whether a system of differential equations arises as the Euler–Lagrange equations of some Lagrangian. This problem has been studied and solved by several authors using different techniques, among them we may cite [1,2,6,19,22]. New ways of attacking these problems have recently appeared, see [16,23,24].

3.1. Main definitions and properties

In this section we will define and study first order local variational problems in the framework of the inverse problem of the calculus of variations. Our basic entities will be Lagrangian densities rather than differential equations. This approach will allow us to give an equivalent formulation of the inverse problem solely in terms of Lagrangian densities.

We start with the introduction of variational problems defined by local data. Let $\mathcal{L} = \{L_\alpha \in \mathcal{L}ag_Y(U_\alpha)\}_{\alpha\in I}$ be a family of local sections of $\mathcal{L}ag_Y$ subordinate to an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ of Y. We denote by $E_{\mathcal{L}} = \{E(L_\alpha)\}_{\alpha\in I}$ the family of local Euler–Lagrange forms obtained by applying the Euler–Lagrange morphism to the family $\mathcal{L}$.

Definition 1. We shall say that $\{\mathfrak{U}, \mathcal{L}\}$ are the data of a local variational problem if the family $E_{\mathcal{L}} = \{E(L_\alpha)\}_{\alpha\in I}$ defines a global Euler–Lagrange form, that is

$$E(L_\alpha)_{|U_\alpha\cap U_\beta} = E(L_\beta)_{|U_\alpha\cap U_\beta}, \quad \forall\, \alpha, \beta \in I.$$

Two local variational data $\{\mathfrak{U}, \mathcal{L}\}$, $\{\mathfrak{U}', \mathcal{L}'\}$ are equivalent if they define the same Euler–Lagrange form $E_{\mathcal{L}} = E_{\mathcal{L}'}$. A local variational problem is an equivalence class $[\{\mathfrak{U}, \mathcal{L}\}]$ of local variational data.

A local variational problem $[\{\mathfrak{U}, \mathcal{L}\}]$ is termed global in case there exists a global Lagrangian density $L \in H^0(Y, \mathcal{L}ag_Y)$ such that $E(L) = E_{\mathcal{L}}$.

Given local variational data $\{\mathfrak{U}, \mathcal{L}\}$ it is clear that the extremals of $L_{\alpha_{|U_\alpha\cap U_\beta}}$ and $L_{\beta_{|U_\alpha\cap U_\beta}}$ coincide, allowing us to give a coherent definition of the global extremals. Moreover, it is clear that the extremality condition only depends on the equivalence class $[\{\mathfrak{U}, \mathcal{L}\}]$. Therefore, we may give the following definition.

Definition 2. A local section s of $\pi : Y \to X$ is critical for the local variational problem $[\{\mathfrak{U}, \mathcal{L}\}]$ if, for every $\alpha \in I$, $s_{|U_\alpha}$ is critical for the variational problem defined by L_α.

The characterizations of the critical sections that we have seen at the end of the previous section imply the following proposition.

Proposition 1. *A local section s is critical for a local variational problem* $[\{\mathfrak{U}, \mathcal{L}\}]$ *if and only if*

$$E_{\mathcal{L}} \circ j^2 s = 0.$$

Equivalently, s is critical if and only if

$$(i_D\,\mathrm{d}\Theta_\alpha)_{|_{j^1 s}} = 0, \quad \forall\, \alpha \in I,\ \ \forall\, D \in \mathfrak{X}(J^1 Y).$$

Let $\mathcal{L} = \{L_\alpha \in \mathcal{L}ag_Y(U_\alpha)\}_{\alpha\in I}$ be a family of sections of $\mathcal{L}ag_Y$ subordinate to an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ of Y. We denote by $\Sigma_{\mathcal{L}} = \{\mathrm{d}\Theta_{L_\alpha}\}_{\alpha\in I}$ the family of differentials of the local Poincaré–Cartan forms. As a consequence of the first variation formula we have the following proposition.

Proposition 2. *If the family* $\Sigma_{\mathcal{L}} = \{\mathrm{d}\Theta_{L_\alpha}\}_{\alpha\in I}$ *defines a global* $(n+1)$*-differential form, that is*

$$(\mathrm{d}\Theta_{L_\alpha})_{|U_\alpha\cap U_\beta} = (\mathrm{d}\Theta_{L_\beta})_{|U_\alpha\cap U_\beta}, \quad \forall\, \alpha, \beta \in I,$$

then $\{\mathfrak{U}, \mathcal{L}\}$ *are the data of a local variational problem.*

Proof. If $(\mathrm{d}\Theta_{L_\alpha})_{|U_\alpha\cap U_\beta} = (\mathrm{d}\Theta_{L_\beta})_{|U_\alpha\cap U_\beta}$ the first variation formula implies the equality

$$E(L_\alpha)_{|U_\alpha\cap U_\beta} - E(L_\beta)_{|U_\alpha\cap U_\beta} = d_v[(\theta \wedge \Omega_{L_\alpha})_{|U_\alpha\cap U_\beta} - (\theta \wedge \Omega_{L_\beta})_{|U_\alpha\cap U_\beta}].$$

The left hand side of this equation is a one-contact, $(n-1)$-horizontal differential form, whereas the right hand side is a two-contact, $(n-1)$-horizontal differential form. Therefore both sides of the equality must vanish identically. □

We can now give the following definition.

Definition 3. We shall say that $\{\mathfrak{U}, \mathcal{L}\}$ are the data of a local variational problem of symplectic type if the family $\Sigma_{\mathcal{L}} = \{\mathrm{d}\Theta_{L_\alpha}\}_{\alpha\in I}$ defines a global differential form which will be called the (formal) symplectic form.

Two data of symplectic type $\{\mathfrak{U}, \mathcal{L}\}$, $\{\mathfrak{U}', \mathcal{L}'\}$ are equivalent if $\Sigma_{\mathcal{L}} = \Sigma_{\mathcal{L}'}$. A variational problem of symplectic type is an equivalence class of local variational data of symplectic type.

As we have seen above, given two local variational data $\{\mathfrak{U}, \mathcal{L}\}$, $\{\mathfrak{U}', \mathcal{L}'\}$, the first variation formula implies that $E_{\mathcal{L}} = E_{\mathcal{L}'}$ if and only if $\Sigma_{\mathcal{L}} = \Sigma_{\mathcal{L}'}$. Therefore it follows that two local variational data of symplectic type are equivalent if and only if they are equivalent as ordinary local variational data.

3.2. Global inverse problem

We proceed now to recall the main definitions of the theory that leads to the solution of the global inverse problem of the calculus of variations. We will use the notation and follow the approach of [1,2]. The reader is referred to these papers for a thorough explanation of the concepts used in this section.

Given a p-form ω on $J^k Y$ it is well known that we can define from it a horizontal p-form $\psi_{k+1}(\omega)$ on $J^{k+1} Y$ which is called the horizontalization of ω. The value of $\psi_{k+1}(\omega)$ at a point $j_x^{k+1}s \in J^{k+1}Y$ is

$$(\psi_{k+1}(\omega))(j_x^{k+1}s) = (\pi^{k+1})^*((j^k s^*\omega)(x)),$$

which is well-defined since $(j^k s^*\omega)(x)$ only depends on the $(k+1)$-jet of the local section s at $x \in X$.

Therefore, for every $k \geq 0$ and $0 \leq p \leq \dim X$, we have a morphism of sheaves of graded algebras

$$\psi_{k+1} : (\pi_0^k)_*\Omega^p_{J^kY} \to (\pi_0^{k+1})_*\Omega^{0,p}_{J^{k+1}Y}.$$

The image of the sheaf $(\pi_0^k)_*\Omega^p_{J^kY}$ under ψ_{k+1} will be denoted $\mathcal{J}^p_{k+1}$. In what follows we will also use the notation $\Omega_k^p \equiv (\pi_0^k)_*\Omega^p_{J^kY}$ and $\Omega_k^{p,q} \equiv (\pi_0^k)_*\Omega^{p,q}_{J^kY}$.

Let us recall, see [18], that the pullback of a contact form under the jet prolongation of a local section vanishes. Thus, taking into account the definition of ψ_{k+1}, it is clear that any contact form is in the kernel of the horizontalization morphisms.

In order to see the cohomological obstructions that characterize the global inverse problem we need to introduce two complexes of sheaves. The exterior differential induces a morphism of sheaves

$$D : \mathcal{J}_k^p \to \mathcal{J}_k^{p+1},$$

characterized by the properties $D \circ D = 0$ and $D \circ \psi_k = \psi_k \circ d$. We denote by $(\mathcal{J}_k^\bullet, D)$ the complex of sheaves

$$\mathcal{J}_k^0 \xrightarrow{D} \mathcal{J}_k^1 \xrightarrow{D} \cdots \to \mathcal{J}_k^{n-1} \xrightarrow{D} \Omega_k^{0,n} \xrightarrow{E} \mathcal{E}_{2k} \to 0,$$

where $\mathcal{E}_{2k}$ is the image sheaf of the Euler–Lagrange morphism. Let $(\Omega_k^\bullet, d)$ be the complex

$$\Omega_k^0 \xrightarrow{d} \Omega_k^1 \xrightarrow{d} \cdots \to \Omega_k^{n-1} \xrightarrow{d} \Omega_k^n \xrightarrow{d} Z_k^{n+1} \to 0,$$

where $Z_k^{n+1} \equiv \mathrm{Ker}(\Omega_k^{n+1} \xrightarrow{d} \Omega_k^{n+2})$.

It is a classical fact that the Poincaré lemma implies that the complex $(\Omega_k^\bullet, d)$ is a resolution of the constant sheaf $\mathbb{R}$ on Y. However, it is a deep result, proved in [1], that the complex $(\mathcal{J}_k^\bullet, D)$ is also a resolution of the same sheaf. Moreover, there exists a unique map $\chi_k : Z_{k-1}^{n+1} \to \mathcal{E}_{2k}$ such that the following diagram is commutative:

$$\begin{array}{ccccccccccccccc}
0 & \to & \mathbb{R} & \to & \Omega_{k-1}^0 & \xrightarrow{d} & \Omega_{k-1}^1 & \xrightarrow{d} & \cdots & \to & \Omega_{k-1}^{n-1} & \xrightarrow{d} & \Omega_{k-1}^n & \xrightarrow{d} & Z_{k-1}^{n+1} & \to 0 \\
 & & \| & & \downarrow \psi_k & & \downarrow \psi_k & & & & \downarrow \psi_k & & \downarrow \psi_k & & \downarrow \chi_k & \\
0 & \to & \mathbb{R} & \to & \mathcal{J}_k^0 & \xrightarrow{D} & \mathcal{J}_k^1 & \xrightarrow{D} & \cdots & \to & \mathcal{J}_k^{n-1} & \xrightarrow{D} & \Omega_k^{0,n} & \xrightarrow{E} & \mathcal{E}_{2k} & \to 0
\end{array}$$

That is, we have a morphism of resolutions $\psi_k : (\Omega_{k-1}^\bullet, d) \to (\mathcal{J}_k^\bullet, D)$. These morphisms are compatible with the natural inclusions $(\Omega_k^\bullet, d) \hookrightarrow (\Omega_{k+1}^\bullet, d)$ induced by pulling back forms. Therefore one has a commutative diagram of complexes

$$
\begin{array}{ccccccc}
(\Omega_0^\bullet, d) & \hookrightarrow & (\Omega_1^\bullet, d) & \hookrightarrow \cdots \hookrightarrow & (\Omega_{k-1}^\bullet, d) & \hookrightarrow \cdots \\
\downarrow \psi_1 & & \downarrow \psi_2 & & \downarrow \psi_k & \\
(\mathcal{J}_1^\bullet, D) & \hookrightarrow & (\mathcal{J}_2^\bullet, D) & \hookrightarrow \cdots \hookrightarrow & (\mathcal{J}_k^\bullet, D) & \hookrightarrow \cdots
\end{array}
$$

An important consequence of this diagram, which follows from the abstract De Rham theorem, is that the cohomology $H^\bullet(Y, \mathbb{R})$ can be computed by means of any of the resolutions $(\Omega_k^\bullet, d)$.

Let us consider the exact sequence $0 \to \mathcal{H}_k \to \Omega_k^n \xrightarrow{E} \mathcal{E}_{2k} \to 0$, where $\mathcal{H}_k$ is the kernel of the Euler–Lagrange morphism E. We will denote by $\delta : H^0(Y, \mathcal{E}_{2k}) \to H^1(Y, \mathcal{H}_k)$ the connecting homomorphism of the long exact sequence of cohomology associated with this short exact sequence. The following theorem, proved in [1], characterizes the cohomological obstructions that appear in the global inverse problem of the calculus of variations.

Theorem 1. *Let $T \in H^0(Y, \mathcal{E}_{2k})$ be a kth order, locally variational operator on Y:*

(i) *T is globally variational, that is $T = E(L)$ for some $L \in H^0(J^kY, \Omega^{0,n}_{J^kY})$, if and only if $\delta(T) = 0$.*

(ii) *Associated to each $T \in H^0(Y, \mathcal{E}_{2k})$ there is a closed $(n+1)$-form ω_T in J^kY such that $\delta(T) = 0$ if and only if the cohomology class $[\omega_T] \in H^{n+1}(J^kY, \mathbb{R})$ vanishes. More concretely, there is a commutative diagram*

$$
\begin{array}{ccccc}
H^0(J^kY, Z_k^{n+1}) & \xrightarrow{[\ ]} & H^{n+1}(J^kY, \mathbb{R}) & \longrightarrow & 0 \\
\chi_{k+1} \downarrow & & \wr \downarrow \chi_{k+1*} & & \\
H^0(Y, E(\mathcal{J}_{k+1}^n)) & \xrightarrow{\delta} & H^1(Y, \mathcal{H}_{k+1}) & \longrightarrow & 0
\end{array}
$$

where $E(\mathcal{J}^n_{k+1}) \supset \mathcal{E}_{2k}$ is the image of the sheaf $\mathcal{J}^n_{k+1}$ under the Euler–Lagrange morphism, and [ ] is the map that takes a closed form to its cohomology class.

3.3. Cohomological obstructions for local variational problems of symplectic type

We now apply these techniques to prove our main result in the case of local variational problems of symplectic type. That is, we give an explicit expression for the cohomological obstruction which characterizes whether a local variational problem of this type is global.

Theorem 2. *Let $\{\mathfrak{U}, \mathcal{L}\}$ be the data of a local variational problem of symplectic type. Then*

1. *$\Sigma_{\mathcal{L}} \in Z^{n+1}_{DR}(J^1Y)$, where $Z^{n+1}_{DR}(J^1Y)$ denotes the De Rham $(n+1)$-cocycles.*
2. *The variational problem $[\{\mathfrak{U}, \mathcal{L}\}]$ is global if and only if the cohomology class $[\Sigma_{\mathcal{L}}] \in H^{n+1}(J^1Y, \mathbb{R})$ vanishes.*
3. *If $[\Sigma_{\mathcal{L}}] = 0$ then there exists $L \in H^0(Y, \mathcal{L}ag_Y)$ such that $E(L) = E_{\mathcal{L}}$ and $\Sigma_{\mathcal{L}} = \mathrm{d}\Theta_L$.*

Proof. The first part is obvious from the definition of $\Sigma_{\mathcal{L}}$.

We now proceed to prove the second assertion. It is clear that the Euler–Lagrange form $E_{\mathcal{L}} \in H^0(Y, \mathcal{E}_{2k})$ is a locally variational operator. Therefore the hypotheses of Theorem 1 hold.

Let us consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_1^n & \longrightarrow & \Omega_1^n & \xrightarrow{d} & Z_1^{n+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \chi_2} & & \\ 0 & \longrightarrow & \mathcal{H}_2 & \longrightarrow & \Omega_2^{0,n} & \xrightarrow{E} & \mathcal{E}_4 & \longrightarrow & 0 \end{array}$$

The pullback of forms $(\pi^4_2)^* : \Omega_2^{1,n} \to \Omega_4^{1,n}$ induces a natural inclusion of $\mathcal{E}_2$ in $\mathcal{E}_4$. Thus the obstruction class $\delta(E_{\mathcal{L}})$ for the existence of a global Lagrangian is determined by the cohomology class of any $\omega \in H^0(Y, Z_1^{n+1})$ such that $\chi_2(\omega) = (\pi^4_2)^*(E_{\mathcal{L}})$. Since Z_1^{n+1} is a quotient sheaf, to give ω is equivalent to finding an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ of Y and differential forms $\Theta_\alpha \in \Omega_1^n(U_\alpha)$ such that

$$\omega_{|U_\alpha} = \mathrm{d}\Theta_\alpha, \quad \forall\, \alpha \in I,$$

and

$$(\mathrm{d}\Theta_\alpha)_{|U_\alpha\cap U_\beta} = (\mathrm{d}\Theta_\beta)_{|U_\alpha\cap U_\beta}, \quad \forall\, \alpha, \beta \in I.$$

The condition $\chi_2(\omega) = (\pi^4_2)^*(E_{\mathcal{L}})$ is fulfilled if and only if

$$E(\psi_2(\Theta_\alpha)) = (\pi^4_2)^*(E_{\mathcal{L}|U_\alpha}).$$

If we take as Θ_α the local Poincaré–Cartan form $\Theta_{L_\alpha} = L_\alpha - \theta \wedge \Omega_{L_\alpha}$ it is clear that the family $\{\Theta_{L_\alpha}\}_{\alpha\in I}$ fulfills the second condition. On the other hand, L_α is horizontal and the horizontalization of $\theta \wedge \Omega_{L_\alpha}$ vanishes since it is a one-contact, n-horizontal differential form, hence

$$\psi_2(\Theta_{L_\alpha}) = \psi_2(L_\alpha) = (\pi^2_1)^*(L_\alpha),$$

which in turn implies

$$E(\psi_2(\Theta_{L_\alpha})) = E((\pi^2_1)^* L_\alpha) = (\pi^4_2)^*(E(L_\alpha))$$

as required. Hence we may take $\omega = \Sigma_{\mathcal{L}}$ as we wanted to prove.

The last part of the theorem follows immediately from the previous ones. □

Remark 1. Let us note that since J^1Y is a deformation retract of Y the obstruction to the existence of a global Lagrangian lives in $H^{n+1}(Y, \mathbb{R})$.

If we take into account Proposition 1 we can now characterize the critical sections in terms of $\Sigma_{\mathcal{L}}$.

Proposition 3. *A local section s is critical for a local variational problem* $[\{\mathfrak{U}, \mathcal{L}\}]$ *of symplectic type if and only if*

$$(i_D \Sigma_{\mathcal{L}})_{|_{j^1 s}} = 0, \quad \forall\, D \in \mathfrak{X}(J^1Y).$$

4. Chern–Simons theory

Let G be an arbitrary Lie group and let $\langle \#, \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$ be an Ad-invariant metric in the Lie algebra of G, and let ω be a connection on the G-bundle $\pi : P \to X$ with curvature Ω^ω. Let $\langle \Omega^\omega \wedge \Omega^\omega \rangle$ be the Chern–Weil four-form associated with the metric $\langle \#, \rangle$. This form is the lifting of a four-form on X that we continue to denote by $\langle \Omega^\omega \wedge \Omega^\omega \rangle$. The Chern–Simons form is a primitive of $\langle \Omega^\omega \wedge \Omega^\omega \rangle$ on P, more precisely we have the following definition.

Definition 4. The Chern–Simons form of the connection ω is the three-form

$$\mathrm{CS}(\omega) = \langle \omega \wedge \Omega^\omega \rangle - \tfrac{1}{6} \langle \omega \wedge [\omega, \omega] \rangle \in \Omega^3(P).$$

We now state the principal properties of the Chern–Simons form; the proofs can be found in [8].

Proposition 4. *The Chern–Simons form has the following properties*:

1. $\mathrm{dCS}(\omega) = \langle \Omega^\omega \wedge \Omega^\omega \rangle$.
2. $R_g^* \mathrm{CS}(\omega) = \mathrm{CS}(\omega)$, *for every* $g \in G$.
3. *If* $\varphi : P \to P$ *is a gauge transformation with associated mapping* $\hat{\varphi} : P \to G$, *then*

$$\varphi^* \mathrm{CS}(\omega) = \mathrm{CS}(\omega) + \mathrm{d}\langle Ad_{\hat{\varphi}^{-1}}\, \omega \wedge \hat{\theta} \rangle - \tfrac{1}{6} \langle \hat{\theta} \wedge [\hat{\theta}, \hat{\theta}] \rangle,$$

where $\hat{\theta} = \hat{\varphi}^* \theta$ *and* θ *is the Maurer–Cartan form of* G.

4.1. The bundle of connections and the Chern–Simons morphisms

The bundle of connections $C(P)$ of a principal bundle $P \to X$ was introduced for the first time in the paper [9] where its basic properties were studied. The natural symplectic structure of $C(P)$ was introduced later in [11]. A more recent exposition of the geometry of $C(P)$ with a particular study of the symplectic structure and its attached Hamiltonian structure can be found in [5].

In order to establish our notation let us recall the definition and main properties of the bundle of connections. The reader may consult [5] for further details.

Let us recall that given a principal G-bundle $\pi : P \to X$ one has the so called Atiyah sequence, see [3], which is the exact sequence of vector bundles over X

$$0 \to \mathrm{ad}\, P \to T_G P \xrightarrow{\pi_*} TX \to 0,$$

where ad P is the adjoint bundle and $T_G P$ is the vector bundle obtained as the quotient of TP under the natural action of the Lie group G.

There exists a natural bijection between connections on P and splittings of the Atiyah sequence. Therefore the bundle of connections $\bar{\pi} : C(P) \to X$ is defined as the affine sub-bundle of $\mathrm{Hom}(TX, T_G P)$ modeled on the vector bundle $\mathrm{Hom}(TX, \mathrm{ad}\, P)$ and determined by all the $\mathbb{R}$-linear mappings $\omega_x : T_x X \to T_G P_x$ such that $\pi_* \circ \omega_x = \mathrm{Id}_{T_x X}$.

Let us denote by $\bar{\pi}^1_0 : J^1C(P) \to C(P)$ the one-jet bundle of $\bar{\pi} : C(P) \to X$. Let $U \subset X$ be a trivializing open set of P with trivialization $\varphi_U : P_U \tilde{\to} U \times G$ and let $s_U : U \to P_U$ be its corresponding local section. We denote also $C(P)_U = (\bar{\pi})^{-1}(U)$.

Definition 5. The Chern–Simons morphism associated with the trivialization $\varphi_U : P_U \tilde{\to} U \times G$ is the mapping

$$J^1C(P)|_{C(P)_U} \xrightarrow{\mathrm{CS}_U} \bigwedge^3 T^*X_U,$$

defined by $\mathrm{CS}_U(j^1_x\omega) = s^*_U(\mathrm{CS}(\omega_x)) \in \bigwedge^3 T^*_xX$.

Remark 2. $\mathrm{CS}_U : J^1C(P)|_{C(P)_U} \to \bigwedge^3 T^*X_U$ is a fibered morphism over $U \subset X$. Thus, CS_U is a local section of the sheaf of Lagrangians $\mathcal{L}ag_{C(P)}$.

In order to give the local expression of a Chern–Simons morphism, let us remember that if $\varphi_U : P_U \tilde{\to} U \times G$ is a local trivialization on an open set $U \subset X$ endowed with local coordinates $\{x^i\}$, and D_α are the G-invariant vector fields on P_U defined by a basis $\mathcal{B} = \{B_1, \ldots, B_m\}$ of the Lie algebra $\mathfrak{g}$, then we can define on $C(P)_U$ the functions A^α_i by means of

$$H_\omega\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial x^i} - A^\alpha_i(\omega)D_\alpha, \quad \omega \in \Gamma(U, C(P)_U),$$

where H_ω is the horizontal lift associated with the connection ω. Then, see [11], the functions $\{x^i, A^\alpha_i\}$ define a local fibered coordinate system on $C(P)_U$. We shall denote by $\{x^i, A^\alpha_i, A^\alpha_{i,j}\}$ the coordinate system induced on $J^1C(P)|_{C(P)_U}$.

From now on we suppose that X is a manifold of dimension 3.

Proposition 5. *The local expression of* CS_U, *with respect to the coordinate system* $\{x^i, A^\alpha_i, A^\alpha_{i,j}\}$, *is* $\mathrm{CS}_U = \mathcal{L}^{\mathrm{CS}}_U \cdot \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ *with*

$$\mathcal{L}^{\mathrm{CS}}_U = \epsilon^{ijk}(A^\alpha_i A^\beta_{k,j} + \tfrac{1}{3}C^\beta_{\mu\nu}A^\alpha_i A^\mu_j A^\nu_k)\langle B_\alpha \otimes B_\beta\rangle,$$

where $C^\beta_{\mu\nu}$ *are the structure constants of the Lie algebra* $\mathfrak{g}$ *with respect to the basis* $\mathcal{B}$.

Proof. Let us denote by $\{\omega^\alpha\}$ the dual basis of the basis of G-invariant vector fields $\{D_\alpha\}$. Now the expression of a connection on P_U is $\omega = (A^\alpha_i\,\mathrm{d}x^i + \omega^\alpha) \otimes D_\alpha$, where, for shortness, we have written A^α_i instead of $A^\alpha_i(\omega)$. Therefore

$$\mathrm{CS}_U \circ j^1\omega = \langle \omega_U \wedge \Omega^\omega_U\rangle - \tfrac{1}{6}\langle \omega_U \wedge [\omega_U, \omega_U]\rangle$$

with $\omega_U = s^*_U\omega \in \Omega^1(U, \mathfrak{g})$. Taking into account that $\omega_U = s^*_U\omega = A^\alpha_i\,\mathrm{d}x^i \otimes B_\alpha$, one has $\Omega^\omega_U = (1/2)F^\alpha_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \otimes B_\alpha$ with $F^\alpha_{ij} = A^\alpha_{j,i} - A^\alpha_{i,j} + C^\alpha_{\mu\nu}A^\mu_i A^\nu_j$. If we substitute this result in the expression of $\mathrm{CS}_U \circ j^1\omega$ we obtain

$$\begin{aligned}\mathrm{CS}_U \circ j^1\omega &= \langle (A_i^\alpha\, \mathrm{d}x^i \otimes B_\alpha) \wedge ((1/2)(A_{k,j}^\beta - A_{j,k}^\beta + C_{\mu\nu}^\beta A_j^\mu A_k^\nu)\, \mathrm{d}x^j \wedge \mathrm{d}x^k \otimes B_\beta)\rangle \\ &\quad - \tfrac{1}{6}\langle (A_i^\alpha\, \mathrm{d}x^i \otimes B_\alpha) \wedge [(A_j^\mu\, \mathrm{d}x^j \otimes B_\mu), (A_k^\nu\, \mathrm{d}x^k \otimes B_\nu)] \\ &= (A_i^\alpha A_{k,j}^\beta + \tfrac{1}{3} C_{\mu\nu}^\beta A_i^\alpha A_j^\mu A_k^\nu)\langle B_\alpha \otimes B_\beta\rangle\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k,\end{aligned}$$

which finishes the proof. □

We are interested now in comparing the Chern–Simons morphisms associated with two different local trivializations of P. As a consequence of part 3 of Proposition 4 one has the following proposition.

Proposition 6. *Let φ_U and φ_V be two local trivializations of $\pi : P \to X$. For every $\omega \in \Gamma(U \cap V, C(P))$ one has*

$$(\mathrm{CS}_V - \mathrm{CS}_U) \circ j^1\omega = \mathrm{d}\langle Ad_{g_{UV^{-1}}}\omega_U \wedge \theta_{UV}\rangle - \tfrac{1}{6}\langle \theta_{UV} \wedge [\theta_{UV}, \theta_{UV}]\rangle,$$

where g_{UV} is the transition function, $\theta_{UV} = g_{UV}^*\theta$ and θ is the Maurer–Cartan form of G.

Corollary 1. *On $C(P)_{U\cap V}$, with respect to the coordinate system $\{x^i, A_i^\alpha, A_{i,j}^\alpha\}$, one has*

$$\mathcal{L}_V^{\mathrm{CS}} - \mathcal{L}_U^{\mathrm{CS}} = \epsilon^{ijk}\left(A_{i,k}^\mu \rho_\mu^\alpha \theta_j^\beta + A_i^\mu \frac{\partial}{\partial x^k}(\rho_\mu^\alpha \theta_j^\beta) - \frac{1}{6} C_{\mu\nu}^\beta \theta_i^\alpha \theta_j^\mu \theta_k^\nu\right)\langle B_\alpha \otimes B_\beta\rangle,$$

where θ_i^α, $\rho_\mu^\alpha \in C^\infty(U \cap V)$ are the functions determined by $\theta_{UV} = \theta_i^\alpha\, \mathrm{d}x^i \otimes B_\alpha$ and $Ad_{g_{UV^{-1}}}(B_\mu) = \rho_\mu^\alpha B_\alpha$, respectively.

4.2. Chern–Simons theory as a local variational problem

In what follows we will see that the Euler–Lagrange operators associated with the Chern–Simons morphisms agree in the intersection of their domains of definition.

Lemma 1. *Let $E : \mathcal{L}ag_{C(P)} \to (\bar{\pi}_0^2)_*\Omega^{1,3}_{J^2C(P)}$ be the Euler–Lagrange morphisms. On $C(P)_{U\cap V}$ one has*

$$E(\mathrm{CS}_U) = E(\mathrm{CS}_V).$$

Proof. Let $L = \mathrm{CS}_V - \mathrm{CS}_U$, then $E(\mathrm{CS}_V) - E(\mathrm{CS}_U) = E(L)$. In the coordinate system $\{x^i, A_i^\alpha, A_{i,j}^\alpha\}$, one has $L = \mathcal{L}\eta$, with $\mathcal{L} = \mathcal{L}_V^{\mathrm{CS}} - \mathcal{L}_U^{\mathrm{CS}}$ and $\eta = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. The expression of the Euler–Lagrange operator in this system of coordinates is

$$E(L) = \left\{\frac{\partial\mathcal{L}}{\partial A_i^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^k}\left(\frac{\partial\mathcal{L}}{\partial A_{i,k}^\alpha}\right)\right\}\mathrm{d}A_i^\alpha \wedge \eta.$$

But if we take into account Corollary 1, we obtain

$$\frac{\partial \mathcal{L}}{\partial A_i^\alpha} = \epsilon^{ijk} \frac{\partial}{\partial x^k}(\rho_\alpha^\mu \theta_j^\nu)\langle B_\mu \otimes B_\nu \rangle,$$
$$\frac{\mathrm{d}}{\mathrm{d}x^k}\left(\frac{\partial \mathcal{L}}{\partial A_{i,k}^\alpha}\right) = \frac{\mathrm{d}}{\mathrm{d}x^k}(\epsilon^{ijk}\rho_\alpha^\mu \theta_j^\nu \langle B_\mu \otimes B_\nu \rangle) = \epsilon^{ijk} \frac{\partial}{\partial x^k}(\rho_\alpha^\mu \theta_j^\nu)\langle B_\mu \otimes B_\nu \rangle,$$

thus $E(L) = 0$ and we conclude the proof. □

According to the definition of a local variational problem given in Definition 1 we have thus proved the following proposition.

Proposition 7. *Let* $\mathfrak{U}_X = \{U_\alpha\}_{\alpha \in I}$ *be a cover of* X *by open trivializing sets of* $\pi : P \to X$*, and let* $\mathfrak{U} = \bar{\pi}^{-1}(\mathfrak{U}_X)$ *be the corresponding cover of* $C(P)$*. Let* $\mathcal{L}^{\mathrm{CS}} = \{\mathrm{CS}_{U_\alpha}\}_{\alpha \in I}$*, one has*

1. $\{\mathfrak{U}, \mathcal{L}^{\mathrm{CS}}\}$ *are the data of a local variational problem.*
2. *The class* $\mathrm{CS}(P) = [\{\mathfrak{U}, \mathcal{L}^{\mathrm{CS}}\}]$ *does not depend on the chosen cover* $\mathfrak{U}_X$*.*

Hence, we can give the following definition.

Definition 6. We shall say that $\mathrm{CS}(P)$ is the local variational problem for the Chern–Simons theory of the bundle $\pi : P \to X$.

Moreover, one can prove the following theorem.

Theorem 3. *The local variational problem* $\mathrm{CS}(P)$ *is of symplectic type.*

Proof. We will use the same notation as in the proof of Lemma 1. Thus, it is enough to prove that $\mathrm{d}\Theta_L = 0$. But, with respect to the local coordinate system, one has

$$\begin{aligned}\Theta_L &= L + \frac{\partial \mathcal{L}}{\partial A_{i,k}^\alpha}(\mathrm{d}A_i^\alpha - A_{i,r}^\alpha\,\mathrm{d}x^r) \wedge i_{\partial/\partial x^k}\eta \\ &= L + \epsilon^{ijk}\rho_\alpha^\mu \theta_j^\nu \langle B_\mu \otimes B_\nu \rangle(\mathrm{d}A_i^\alpha - A_{i,r}^\alpha\,\mathrm{d}x^r) \wedge i_{\partial/\partial x^k}\eta.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{d}\Theta_L &= \mathrm{d}L + \epsilon^{ijk}\langle B_\mu \otimes B_\nu \rangle\{\mathrm{d}(\rho_\alpha^\mu \theta_j^\nu) \wedge (\mathrm{d}A_i^\alpha - A_{i,r}^\alpha\,\mathrm{d}x^r) \wedge i_{\partial/\partial x^k}\eta - \rho_\alpha^\mu \theta_j^\nu\,\mathrm{d}A_{i,k}^\alpha \wedge \eta\} \\ &= \mathrm{d}L - \epsilon^{ijk}\langle B_\mu \otimes B_\nu \rangle\left\{\frac{\partial}{\partial x^k}(\rho_\alpha^\mu \theta_j^\nu)\,\mathrm{d}A_i^\alpha + \rho_\alpha^\mu \theta_j^\nu\,\mathrm{d}A_{i,k}^\alpha\right\} \wedge \eta.\end{aligned}$$

But

$$\mathrm{d}L = \mathrm{d}\mathcal{L} \wedge \eta = \epsilon^{ijk}\langle B_\mu \otimes B_\nu \rangle\left\{\rho_\alpha^\mu \theta_j^\nu\,\mathrm{d}A_{i,k}^\alpha + \frac{\partial}{\partial x^k}(\rho_\alpha^\mu \theta_j^\nu)\,\mathrm{d}A_i^\alpha\right\} \wedge \eta,$$

therefore $\mathrm{d}\Theta_L = 0$. □

The bundle of connections $C(P)$ has a symplectic structure $\Omega_2 \in \Omega^2(C(P), \bar{\pi}^*\mathrm{ad}\, P)$ with values in the adjoint bundle ad P, which is just the two-form induced on $C(P)$ by the curvature two-form of the canonical connection on $J^1P \to C(P)$ by means of the identification $C(P) \simeq J^1P/G$, see [5,11]. Therefore, if $\varphi_U : P_U \xrightarrow{\sim} U \times G$ is a local trivialization of P and U is coordinated by $\{x^i\}$, the expression of Ω_2 in the coordinate system $\{x^i, A_i^\alpha\}$ induced on $C(P)_U$ is

$$\Omega_2 = (\mathrm{d}A_i^\alpha \wedge \mathrm{d}x^i + \tfrac{1}{2}C_{\mu\nu}^\alpha A_j^\mu A_k^\nu\, \mathrm{d}x^j \wedge \mathrm{d}x^k) \otimes D_\alpha,$$

where $\{D_\alpha\}$ is the basis of G-invariant vector fields on P_U, and hence a basis of $\Gamma(C(P)_U, \bar{\pi}^*\mathrm{ad}\, P)$, associated with the basis of the Lie algebra $\mathfrak{g}$ chosen to construct the coordinate system $\{x^i, A_i^\alpha\}$. For further details see [5,11] and Section 6 of this paper.

On the other hand, the curvature morphism is the fibered mapping over X

$$\Omega : J^1C(P) \to \bigwedge^2 T^*X \otimes \mathrm{ad}\, P$$

given by $\Omega(j_x^1\omega) = (\Omega^\omega)_x$. Hence its local expression is

$$\Omega = \tfrac{1}{2}F_{ij}^\alpha\, \mathrm{d}x^i \wedge \mathrm{d}x^j \otimes D_\alpha = \tfrac{1}{2}(A_{j,i}^\alpha - A_{i,j}^\alpha + C_{\mu\nu}^\alpha A_i^\mu A_j^\nu)\, \mathrm{d}x^i \wedge \mathrm{d}x^j \otimes D_\alpha.$$

The symplectic structure Ω_2 of $C(P)$ and the curvature morphism Ω will allow us to intrinsically express the Euler–Lagrange form $E_{\mathcal{L}^{\mathrm{CS}}} \in \Omega^4(J^2C(P))$ and the form $\Sigma_{\mathcal{L}^{\mathrm{CS}}} \in \Omega^4(J^1C(P))$ associated with the local variational problem of symplectic type CS(P).

Theorem 4. *One has*

$$E_{\mathcal{L}^{\mathrm{CS}}} = 2(\bar{\pi}_1^2)^*\langle(\bar{\pi}_0^1)^*\Omega_2 \wedge \Omega\rangle, \qquad \Sigma_{\mathcal{L}^{\mathrm{CS}}} = (\bar{\pi}_0^1)^*\langle\Omega_2 \wedge \Omega_2\rangle.$$

That is, $\Sigma_{\mathcal{L}^{\mathrm{CS}}}$ is the pullback to $J^1C(P)$ of the four-form induced on the bundle of connections $C(P)$ by the Chern–Weil four-form of the canonical connection.

Proof. Let us choose a local trivialization $\varphi_U : P_U \xrightarrow{\sim} U \times G$. As we have seen in Proposition 5, with respect to the local coordinate system $\{x^i, A_i^\alpha, A_{i,j}^\alpha\}$ one has $\mathrm{CS}_U = \mathcal{L}_U^{\mathrm{CS}} \cdot \eta$ with

$$\mathcal{L}_U^{\mathrm{CS}} = \epsilon^{ijk}(A_i^\alpha A_{k,j}^\beta + \tfrac{1}{3}C_{\mu\nu}^\beta A_i^\alpha A_j^\mu A_k^\nu)\langle B_\alpha \otimes B_\beta\rangle,$$

and $\eta = \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$.

Therefore, we have

$$\begin{aligned} E_{\mathcal{L}^{\mathrm{CS}}_{|U}} &= \left\{ \frac{\partial \mathcal{L}_U^{\mathrm{CS}}}{\partial A_i^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^k}\left(\frac{\partial \mathcal{L}_U^{\mathrm{CS}}}{\partial A_{i,k}^\alpha}\right)\right\} \mathrm{d}A_i^\alpha \wedge \eta \\ &= \epsilon^{ijk}\langle B_\alpha \otimes B_\beta\rangle(A_{k,j}^\beta - A_{j,k}^\beta + C_{\mu\nu}^\beta A_j^\mu A_k^\nu)\, \mathrm{d}A_i^\alpha \wedge \eta \\ &= \langle B_\alpha \otimes B_\beta\rangle F_{jk}^\beta\, \mathrm{d}A_i^\alpha \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \\ &= 2\left\langle (\mathrm{d}A_i^\alpha \wedge \mathrm{d}x^i \otimes D_\alpha) \wedge \left(\frac{1}{2}F_{jk}^\beta\, \mathrm{d}x^j \wedge \mathrm{d}x^k \otimes D_\beta\right)\right\rangle \end{aligned}$$

if we take into account the local expressions of Ω_2 and Ω we finish the proof of the first part.

In the same way

$$\begin{aligned}\Theta_{\mathrm{CS}_U} &= \mathrm{CS}_U + \frac{\partial \mathcal{L}_U^{\mathrm{CS}}}{\partial A^\beta_{k,j}} (\mathrm{d}A^\beta_k - A^\beta_{k,r}\,\mathrm{d}x^r) \wedge i_{\partial/\partial x^j}\eta \\ &= \mathrm{CS}_U + \epsilon^{ijk} \langle B_\alpha \otimes B_\beta\rangle A^\alpha_i (\mathrm{d}A^\beta_k - A^\beta_{k,r}\,\mathrm{d}x^r) \wedge i_{\partial/\partial x^j}\eta \\ &= \mathrm{CS}_U + \epsilon^{ijk} \langle B_\alpha \otimes B_\beta\rangle A^\alpha_i (\mathrm{d}A^\beta_k \wedge i_{\partial/\partial x^j}\eta - A^\beta_{k,j}\eta) \\ &= \epsilon^{ijk} \langle B_\alpha \otimes B_\beta\rangle A^\alpha_i \left(\frac{1}{3} C^\beta_{\mu\nu} A^\mu_j A^\nu_k \eta + \mathrm{d}A^\beta_k \wedge i_{\partial/\partial x^j}\eta \right).\end{aligned}$$

Taking the exterior differential, after a little computation, bearing in mind that $\langle\#,\rangle$ is Ad-invariant, we obtain

$$\begin{aligned}\mathrm{d}\Theta_{\mathrm{CS}_U} &= \langle B_\alpha \otimes B_\beta\rangle\, \mathrm{d}A^\alpha_i \wedge \mathrm{d}x^i \wedge (\mathrm{d}A^\beta_j \wedge \mathrm{d}x^j + C^\beta_{\mu\nu} A^\mu_j A^\nu_k \mathrm{d}x^j \wedge \mathrm{d}x^k) \\ &= \langle (\mathrm{d}A^\alpha_i \wedge \mathrm{d}x^i \otimes D_\alpha) \wedge \{(\mathrm{d}A^\beta_j \wedge \mathrm{d}x^j + C^\beta_{\mu\nu} A^\mu_j A^\nu_k\, \mathrm{d}x^j \wedge \mathrm{d}x^k) \otimes D_\beta\}\rangle,\end{aligned}$$

and the local expression of Ω_2 allows us to finish the proof. □

The expression found for the Euler–Lagrange operator allows us to characterize the critical sections.

Corollary 2. *A connection $\omega \in \Gamma(X, C(P))$ is critical for the variational problem* CS(P) *if and only if it is zero the component of its curvature form corresponding to the subspace of $\mathfrak{g}$ where $\langle\#,\rangle$ is nondegenerate. If $\langle\#,\rangle$ is nondegenerate, ω is critical if and only it is a flat connection.*

The inverse problem of the Calculus of Variations, Theorem 2, allows us to prove the following corollary.

Corollary 3. *There exists a global section $L^{\mathrm{CS}} \in H^0(C(P), \mathcal{L}ag_{C(P)})$ such that $E(L^{\mathrm{CS}}) = E_{\mathcal{L}^{\mathrm{CS}}}$ and $\Sigma_{\mathcal{L}^{\mathrm{CS}}} = \mathrm{d}\Theta_{L^{\mathrm{CS}}}$.*

Proof. Since $\bar{\pi} : C(P) \to X$ is an affine bundle and X has dimension 3 it follows that $H^4(C(P), \mathbb{R}) = 0$, therefore $[\Sigma_{\mathcal{L}^{\mathrm{CS}}}] = 0$. Hence, applying Theorem 2 we obtain the result. □

Remark 3. The local solution of the inverse problem of the calculus of variations provides us with an algorithm which determines only the local expression of a Lagrangian whose Euler–Lagrange equations are the given ones. When applied in this case we recover the Chern–Simons morphism CS_U of the open set in which we are working on. Therefore the inverse problem of the calculus of variations does not help us in finding the explicit expression of L^{CS}. According to the best of our knowledge, the existence of the global Lagrangian L^{CS} is not found in the literature.

5. Infinitesimal symmetries and Noether invariants

The topic that we will address in this section is that of infinitesimal symmetries and Noether invariants of variational problems defined by local data.

5.1. Global variational problems

An infinitesimal symmetry of the first order variational problem defined by a Lagrangian density L is a vector field $D \in \mathfrak{X}(J^1Y)$ such that the Lie derivative of the Poincaré–Cartan form Θ_L with respect to D is an exact form, that is

$$L_D\Theta_L = -\mathrm{d}\eta_D.$$

In that case, we have $i_D\,\mathrm{d}\Theta_L = -\mathrm{d}\omega_D$ with $\omega_D = i_D\Theta_L + \eta_D$. Noether's theorem can now be formulated as follows. Let s be a critical section; taking into account the second characterization of critical sections given in Eq. (3), we have

$$0 = (i_D\,\mathrm{d}\Theta_L)_{|_{j^1s}} = -\mathrm{d}(\omega_D)_{|_{j^1s}}.$$

That is, ω_D is a closed $(n-1)$-form along $j^1s(X) \subset J^1Y$, or what amounts to the same, $(j^1s)^*\omega_D$ is a closed differential form on X.

From its definition it is clear that ω_D is determined up to the addition of any closed $(n-1)$-form on J^1Y. The Noether invariant associated with the infinitesimal symmetry D is the coset of $(n-1)$-differential forms

$$\omega_D = i_D\Theta_L\omega_D + \eta_D + Z_{DR}^{n-1}(J^1Y),$$

where $Z_{DR}^{n-1}(J^1Y)$ is the space of De Rham $(n-1)$-cocycles on J^1Y.

5.2. Local variational problems of symplectic type

In what follows we will consider the problem of defining infinitesimal symmetries and their associated Noether invariants for local variational problems of symplectic type.

Definition 7. Let $\{\mathfrak{U}, \mathcal{L}\}$ be the data of a local variational problem. We shall say that $D \in \mathfrak{X}(J^1Y)$ is an infinitesimal symmetry of the variational problem $[\{\mathfrak{U}, \mathcal{L}\}]$ if, for every $\alpha \in I$, $D_{|_{(\pi_0^1)^{-1}(U_\alpha)}}$ is an infinitesimal symmetry of the variational problem associated with L_α. That is, one has

$$L_D\Theta_\alpha = -\mathrm{d}\eta_D^\alpha$$

with $\Theta_\alpha = \Theta_{L_\alpha}$ and $\eta_D^\alpha \in (\pi_0^1)_*\Omega_{J^1Y}^{n-1}(U_\alpha)$.

The local Noether invariant associated with the infinitesimal symmetry D on U_α is the coset

$$\omega_D^\alpha = i_{\bar{D}}\Theta_\alpha + \eta_D^\alpha + (\pi_0^1)_* Z_{J^1Y}^{n-1}(U_\alpha),$$

where $Z_{J^1Y}^{n-1}$ is the sheaf of De Rham $(n-1)$-cocycles on J^1Y.

One has the following proposition.

Proposition 8. *If $D \in \mathfrak{X}(Y)$ is an infinitesimal symmetry of the local variational problem associated with $\{\mathfrak{U}, \mathcal{L}\}$, then $\{\omega_D^\alpha\}_{\alpha\in I}$ defines an element*

$$[\omega_D] \in H^0\left(J^1Y, \frac{\Omega_{J^1Y}^{n-1}}{Z_{J^1Y}^{n-1}}\right).$$

Proof. It is clear that $i_D\,\mathrm{d}\Theta_\alpha = -\mathrm{d}\omega_D^\alpha$. Since the variational problem is of symplectic type we have

$$(\mathrm{d}\Theta_{L_\alpha})_{|U_\alpha\cap U_\beta} = (\mathrm{d}\Theta_{L_\beta})_{|U_\alpha\cap U_\beta}, \quad \forall\, \alpha, \beta \in I,$$

and this implies

$$\mathrm{d}(\omega_D^\alpha - \omega_D^\beta)_{|U_\alpha\cap U_\beta} = 0.$$

That is $(\omega_D^\alpha - \omega_D^\beta)_{|U_\alpha\cap U_\beta} \in Z_{J^1Y}^{n-1}((\pi_0^1)^{-1}(U_\alpha) \cup (\pi_0^1)^{-1}(U_\alpha))$. Taking into account that $(\pi_0^1)^{-1}\mathfrak{U}$ is a cover of J^1Y the proof is finished. □

If there exists a global Noether invariant then it is clear from the definition that it is defined up to the addition of any closed $(n-1)$-form on J^1Y.

Definition 8. Let $D \in \mathfrak{X}(Y)$ be an infinitesimal symmetry of the local variational problem $[\{\mathfrak{U}, \mathcal{L}\}]$. The element

$$[\omega_D] \in H^0\left(J^1Y, \frac{\Omega_{J^1Y}^{n-1}}{Z_{J^1Y}^{n-1}}\right)$$

is called the virtual Noether invariant associated with D. We shall say that D admits $\omega_D \in H^0(J^1Y, \Omega_{J^1Y}^{n-1})$ as a global Noether invariant if its image in $H^0(J^1Y, \Omega_{J^1Y}^{n-1}/Z_{J^1Y}^{n-1})$ is $[\omega_D]$.

Noether's theorem is also valid for symmetries which admit a global Noether invariant

Theorem 5. *If D is an infinitesimal symmetry with a global Noether invariant ω_D, then for every critical section s one has*

$$\mathrm{d}(\omega_D)_{|_{j^1s}} = 0,$$

thus $(j^1s)^*\omega_D \in Z^{n-1}(X)$.

Proof. On U_α one has $i_{\bar{D}}\,\mathrm{d}\Theta_\alpha = -\mathrm{d}\omega_D^\alpha = -\mathrm{d}\omega_{D_{|U_\alpha}}$. Since s is critical the result follows immediately from Proposition 1. □

The following proposition is important since it allows us to manage infinitesimal symmetries and their Noether invariants in a practical way. First we give a useful representation of the virtual Noether invariant in terms of an ordinary differential form rather than a section of a quotient sheaf. As a consequence we are also able to exhibit an explicitly computable criterion for the existence of global Noether invariants.

Proposition 9. *Let $D \in \mathfrak{X}(Y)$ be an infinitesimal symmetry of the local variational problem $[\{\mathfrak{U}, \mathcal{L}\}]$*

1. *The virtual Noether invariant $[\omega_D]$ associated with D gets identified with $\xi_D = -i_{\bar{D}}\Sigma_{\mathcal{L}} \in H^0(J^1Y, Z^n_{J^1Y}) = Z^n_{DR}(J^1Y)$.*
2. *D admits global Noether invariants if and only if the cohomology class $[i_{\bar{D}}\Sigma_{\mathcal{L}}] \in H^n(J^1Y, \mathbb{R})$ vanishes. That is, if and only if there exists $\omega_D \in \Omega^{n-1}(J^1Y)$ such that $i_{\bar{D}}\Sigma_{\mathcal{L}} = -\mathrm{d}\omega_D$. In that case, the set of global Noether invariants associated with D is the class*

$$\zeta_D = \omega_D + Z^{n-1}_{DR}(J^1Y),$$

that is, $\zeta_D \in \Omega^{n-1}(J^1Y)/Z^{n-1}_{DR}(J^1Y) \simeq B^n_{DR}(J^1Y)$.

Proof. The De Rham resolution on J^1Y gives us the short exact sequence of sheaves

$$0 \to Z^{n-1}_{J^1Y} \to \Omega^{n-1}_{J^1Y} \to Z^n_{J^1Y} \to 0.$$

Therefore $\Omega^{n-1}_{J^1Y}/Z^{n-1}_{J^1Y} \simeq Z^n_{J^1Y}$ and $[\omega_D] = \{\omega_D^\alpha\}_{\alpha\in I}$ gets identified with

$$\{\mathrm{d}\omega_D^\alpha\}_{\alpha\in I} = -\{i_{\bar{D}}\,\mathrm{d}\Theta_\alpha\}_{\alpha\in I} = -i_{\bar{D}}\Sigma_{\mathcal{L}},$$

which proves the first part.

If we take cohomology in the above short exact sequence and we take into account that $\Omega^{n-1}_{J^1Y}$ is an acyclic sheaf, we obtain an exact sequence

$$0 \to H^0(J^1Y, Z^{n-1}_{J^1Y}) \to H^0(J^1Y, \Omega^{n-1}_{J^1Y}) \to H^0(J^1Y, Z^n_{J^1Y}) \xrightarrow{\delta} H^1(J^1Y, Z^{n-1}_{J^1Y}) \to 0.$$

Hence it is clear that D admits a global Noether invariant if and only if $\delta(i_{\bar{D}}\Sigma_{\mathcal{L}}) = 0$. By the abstract De Rham's theorem [25], we have

$$H^1(J^1Y, Z^{n-1}_{J^1Y}) \simeq H^n(J^1Y, \mathbb{R}),$$

and $\delta(i_{\bar{D}}\Sigma_{\mathcal{L}})$ gets identified with the assignment of the cohomology class

$$[i_{\bar{D}}\Sigma_{\mathcal{L}}] \in H^n(J^1Y, \mathbb{R}).$$

The rest of the theorem is a straightforward consequence of the cohomology exact sequence. □

Remark 4. The obstruction to the existence of global Noether invariants lives in $H^n(Y, \mathbb{R})$.

We can now give a cohomological characterization of the infinitesimal symmetries and in particular of those which admit global Noether invariants.

Theorem 6. *Given a vector field* $D \in \mathfrak{X}(J^1Y)$ *one has*

1. *D is an infinitesimal symmetry of the local variational problem* $[\{\mathfrak{U}, \mathcal{L}\}]$ *if and only if*
$$i_D \Sigma_{\mathcal{L}} \in H^0(J^1Y, Z^n_{J^1Y}) = Z^n_{DR}(J^1Y).$$
Equivalently, D is an infinitesimal symmetry if and only if
$$L_D \Sigma_{\mathcal{L}} = 0.$$
2. *D is an infinitesimal symmetry that admits global Noether invariants if and only if the cohomology class* $[i_D \Sigma_{\mathcal{L}}] \in H^n(J^1Y, \mathbb{R})$ *vanishes.*

Proof. If D is an infinitesimal symmetry the previous proposition implies that $i_{\bar{D}} \Sigma_{\mathcal{L}}$ is a closed form. In order to prove the converse statement we can assume, without changing the class of the variational problem $[\{\mathfrak{U}, \mathcal{L}\}]$, that $\mathfrak{U}$ is a good cover of Y. This is not a real restriction since on every (paracompact) manifold we can construct a good cover out of the geodesically convex neighborhoods associated with a Riemannian metric, see [4]. Therefore every open set U_α is contractible and as a consequence there exists ω^α_D such that
$$-\mathrm{d}\omega^\alpha_D = (i_D \Sigma_{\mathcal{L}})_{|U_\alpha} = i_D \,\mathrm{d}\Theta_{L_\alpha},$$
which implies that D is an infinitesimal symmetry. The equivalence with the condition $L_D \Sigma_{\mathcal{L}} = 0$ follows from the fact that $\Sigma_{\mathcal{L}}$ is closed.

The second part of the theorem is a direct consequence of the first one and the preceding proposition. □

We end this section with a theorem that gives the structure of the set of infinitesimal symmetries.

Theorem 7. *One has that*

1. *The set* $\mathcal{D}$ *of infinitesimal symmetries of a local variational problem is a Lie subalgebra of* $\mathfrak{X}(J^1Y)$.
2. *The set* $\mathcal{D}_0$ *of infinitesimal symmetries which admit a global Noether invariant is an ideal of* $\mathcal{D}$, *that is*
$$[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}_0.$$
Moreover, by sending an infinitesimal symmetry D to the cohomology class $[\xi_D] \in H^n(J^1Y, \mathbb{R})$ *of its virtual Noether invariant we get a natural inclusion of the quotient Lie algebra*
$$\frac{\mathcal{D}}{\mathcal{D}_0} \subset H^n(J^1Y, \mathbb{R}) \simeq H^n(Y, \mathbb{R}).$$

Proof. The first statement follows immediately from the previous theorem just by taking into account the well known identity $L_{[D_1,D_2]} = [L_{D_1}, L_{D_2}]$.

On the other hand, if D_1 and D_2 are two infinitesimal symmetries, we have

$$\xi_{[D_1,D_2]} = -i_{[D_1,D_2]}\Sigma_{\mathcal{L}} = -[i_{D_1}, L_{D_2}]\Sigma_{\mathcal{L}} = L_{D_2} i_{D_1}\Sigma_{\mathcal{L}} = \mathrm{d}(i_{D_2} i_{D_1}\Sigma_{\mathcal{L}}).$$

Moreover, it is clear that the sum and product with a scalar preserves $\mathcal{D}_0$. □

Corollary 4. *Let D_1 and D_2 be two infinitesimal symmetries belonging to $\mathcal{D}_0$, then*

$$\omega_{[D_1,D_2]} = i_{D_2} i_{D_1}\Sigma_{\mathcal{L}}$$

is a global Noether invariant for $[D_1, D_2]$.

6. Symplectic structure of $C(P)$ and infinitesimal symmetries of Chern–Simons theories

In this section we will see that the symplectic structure of $C(P)$ allow us to associate to every G-invariant vector field on P an infinitesimal symmetry of the variational problem CS(P). Proving thus, that the Lie algebra of infinitesimal symmetries of Chern–Simons theories is infinite dimensional. In order to see the connection between the symplectic structure of $C(P)$ and the infinitesimal symmetries of Chern–Simons theories we briefly recall the relationship between the bundle of connections $\bar{\pi} : C(P) \to X$ and the one-jet bundle $\pi_0^1 : J^1P \to P$ of the principal bundle $\pi : P \to X$, a complete exposition can be found in [5,9,11].

As is well known [14], every automorphism $\varphi \in \operatorname{Aut} P$ of the principal bundle $\pi : P \to X$ acts on the connections of P, producing a diffeomorphism $\varphi^c : C(P) \to C(P)$. If $D \in \Gamma(X, \operatorname{aut} P)$ is a G-invariant vector field on P, then it is called an infinitesimal automorphism of P, since its uniparametric group $\{\tau_t\}$ acts by automorphisms of P. Thus, $\{\tau_t^c\}$ is a uniparametric group of diffeomorphisms of $C(P)$ whose infinitesimal generator will be denoted by D^c. In this way the natural action of the automorphisms $\operatorname{Aut} P$ of the principal bundle on the connections of P induces a Lie algebra morphism

$$\operatorname{aut} P \to \mathfrak{X}(C(P)), \quad D \to D^c.$$

The group G acts freely on the right on J^1P and the quotient manifold J^1P/G exists and gets identified with the bundle of connections $C(P)$, see [9], in such a way that the following diagram is commutative:

$$\begin{array}{ccc} J^1P & \xrightarrow{q} & C(P) \\ {\scriptstyle \pi_0^1}\downarrow & & \downarrow{\scriptstyle \bar{\pi}} \\ P & \xrightarrow{\pi} & X \end{array}$$

Thus $q : J^1P \to C(P)$ is a principal G-bundle, where q is the natural quotient map. Moreover, J^1P is isomorphic, as a principal G-bundle, to the pullback of P to $C(P)$, that is

$J^1P \simeq \bar{\pi}^*P = C(P) \times_X P$. It follows that we can identify the adjoint bundle ad J^1P of $q : J^1P \to C(P)$ with $\bar{\pi}^*\text{ad}\, P$.

On the other hand, every automorphism $\varphi \in \text{Aut}\, P$ lifts in a natural way to an automorphism $\varphi^1 : J^1P \to J^1P$, and the following diagram is commutative:

$$\begin{array}{ccc} J^1P & \xrightarrow{\varphi^1} & J^1P \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ C(P) & \xrightarrow{\varphi^c} & C(P) \end{array}$$

Moreover, if $D \in \text{aut}\, P$ and we denote by $\bar{D} \in \mathfrak{X}(J^1P)$ its prolongation to J^1P, then $\bar{D}$ is q-projectable and its projection is D^c.

The structure form θ of J^1P turns out to be a principal connection on the bundle $q : J^1P \to C(P)$. θ is called the canonical connection. The symplectic form Ω_2 of $C(P)$ is the two-form with values in ad $J^1P \simeq \bar{\pi}^*\text{ad}\, P$ induced by the curvature Ω^θ of the canonical connection, and every $D \in \text{aut}\, P$ fulfills $L_{\bar{D}}\theta = 0$.

If $\varphi \in \text{Aut}\, P$, then one has $\varphi^{1^*}\theta = \theta$ and thus every $D \in \Gamma(X, \text{aut}\, P)$ satisfies $L_{\bar{D}}\theta = 0$. As the following lemma shows, this property of the G-invariant vector fields will be crucial for obtaining infinitesimal symmetries of Chern–Simons theories.

Lemma 2. *Let $\hat{\pi} : \hat{P} \to \hat{X}$ a principal bundle and let $\hat{\omega}$ be a principal connection on $\hat{P}$. Let $\hat{D} \in \text{aut}\, \hat{P}$ with $\hat{\pi}$-projection $D \in \mathfrak{X}(\hat{X})$. If $L_{\hat{D}}\hat{\omega} = 0$ then*

$$i_D \Omega_{\hat{\omega}} = -d^\nabla(\hat{D}^v),$$

where $\Omega_{\hat{\omega}} \in \Omega^2(\hat{X}, \text{ad}\, \hat{P})$ is the two-form induced by the curvature form of $\hat{\omega}$, $\hat{D}^v$ is the vertical component of the vector field $\hat{D}$ with respect to $\hat{\omega}$ and d^∇ is the covariant differential induced by $\hat{\omega}$ on the sections of ad P.

Proof. On $\hat{P}$ one has

$$\begin{aligned} i_{\hat{D}}\Omega^{\hat{\omega}} &= i_{\hat{D}}(\mathrm{d}\hat{\omega} + \tfrac{1}{2}[\hat{\omega}, \hat{\omega}]) = i_{\hat{D}}\, \mathrm{d}\hat{\omega} + \tfrac{1}{2}[i_{\hat{D}}\hat{\omega}, \hat{\omega}] - \tfrac{1}{2}[\hat{\omega}, i_{\hat{D}}\hat{\omega}]) \\ &= -\mathrm{d}i_{\hat{D}}\hat{\omega} - [\hat{\omega}, i_{\hat{D}}\hat{\omega}], \end{aligned}$$

where we have taken into account that $L_{\hat{D}}\hat{\omega} = 0$. Since $\hat{D}$ is G-invariant $\hat{\omega}(\hat{D}) \in C^\infty(P, \mathfrak{g})$ is Ad-invariant. Therefore it can be identified with a section $\hat{D}^v \in \Gamma(\hat{X}, \text{ad}\, \hat{P})$ which is the vertical component of the vector field $\hat{D}$ with respect to the connection $\hat{\omega}$. Bearing in mind the expression of the covariant derivative in terms of differential forms in $\Omega^\bullet(\hat{P}, \mathfrak{g})_B$, that is the ones which are horizontal and Ad-equivariant, we obtain

$$i_{\hat{D}}\Omega^{\hat{\omega}} = -(\mathrm{d}i_{\hat{D}}\hat{\omega} + [\hat{\omega}, \hat{\omega}(\hat{D})]) = -\mathrm{d}^{\hat{\omega}}(\hat{\omega}(\hat{D})),$$

and we conclude just by taking the one-form on $\hat{X}$ with values in ad $\hat{P}$ that corresponds to it. □

If we apply this lemma to the canonical connection θ on the principal bundle $q : J^1P \to C(P)$ we can prove that for every vector field $D \in \operatorname{aut} P$ the vector field $D^c \in \mathfrak{X}(C(P))$ is an infinitesimal symmetry of the Chern–Simons theory.

Given $D \in \operatorname{aut} P$, we will denote by $\bar{D}^v = \theta(\bar{D})$ the vertical component, with respect to the canonical connection θ, of the one-jet prolongation of the vector field D. Under these conditions we have the following proposition.

Proposition 10. *Let $D \in \operatorname{aut} P$, then*

1. *$D^c \in \mathfrak{X}(C(P))$ is an infinitesimal symmetry of the variational problem $\mathrm{CS}(P)$. That is, D^c belongs to the Lie algebra $\mathcal{D}(\mathrm{CS}(P))$ of the infinitesimal symmetries which are $\bar{\pi}$-projectable.*
2. *D^c admits $\omega_{D^c} = 2(\bar{\pi}_0^1)^*\langle \bar{D}^v \wedge \Omega_2\rangle$ as a global Noether invariant. That is, $D^c \in \mathcal{D}_0(\mathrm{CS}(P))$.*

Therefore we have a Lie algebra morphism

$$\operatorname{aut} P \to \mathcal{D}_0(\mathrm{CS}(P)), \qquad D \to D^c,$$

whose kernel coincides with the center of $\mathfrak{g}$.

Proof. If we denote by $\overline{D^c} \in \mathfrak{X}(J^1C(P))$ the one-jet prolongation of D^c, then we have

$$i_{\overline{D^c}}\Sigma_{\mathcal{L}^{\mathrm{CS}}} = i_{\overline{D^c}}\{(\bar{\pi}_0^1)^*\langle \Omega_2 \wedge \Omega_2\rangle\} = (\bar{\pi}_0^1)^*\{i_{D^c}\langle \Omega_2 \wedge \Omega_2\rangle\} = 2(\bar{\pi}_0^1)^*\{\langle i_{D^c}\Omega_2 \wedge \Omega_2\rangle\}.$$

By the previous lemma $i_{D^c}\Omega_2 = -d^\nabla \bar{D}^v$ and thus

$$i_{\overline{D^c}}\Sigma_{\mathcal{L}^{\mathrm{CS}}} = -2(\bar{\pi}_0^1)^*\{\langle d^\nabla \bar{D}^v \wedge \Omega_2\rangle\} = -2(\bar{\pi}_0^1)^*\,\mathrm{d}\langle \bar{D}^v \wedge \Omega_2\rangle,$$

where the last equality is a consequence of the Bianchi identity $d^\nabla\Omega_2 = 0$. Regarding the last assertion, if $D^c = 0$ it is clear that $D \in \operatorname{aut} P$ is π-vertical.

Finally the proof that the kernel coincides with the center of $\mathfrak{g}$ is rather straightforward, see [11]. □

Remark 5. Therefore, the algebra $\mathcal{D}_0(\mathrm{CS}(P))$ of the infinitesimal symmetries of $\mathrm{CS}(P)$ which admit global Noether invariants is infinite dimensional. Moreover, the Noether invariants associated with the representation of the gauge vector fields are not zero, because if $D \in \operatorname{gau} P$ then $\theta(\bar{D}) \in \operatorname{gau} J^1P$ gets identified with D itself via the natural inclusion $\operatorname{gau} P \hookrightarrow \operatorname{gau} J^1P$ and thus $\bar{D}^v = D$, which implies that the global Noether invariant is $\omega_{D^c} = 2(\bar{\pi}_0^1)^*\langle D \wedge \Omega_2\rangle$.

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